Problem 1  Leibnitz’s Theorem. In calculus you may have learned the one-dimensional form of Leibnitz’s Theorem,
\[ \frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) \, dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} \, dx + \dot{b} f(b(t), t) - \dot{a} f(a(t), t) \]
which shows you how to time-differentiate an integral whose limits are also functions of time. Prove the three-dimensional version,
\[ \frac{d}{dt} \int_{\mathcal{V}(t)} f(x, t) \, dV = \int_{\mathcal{V}(t)} \frac{\partial f}{\partial t} \, dV + \int_{\mathcal{S}(t)} f(x, t)[v \cdot \hat{n}] \, dS, \]
where \( \mathcal{V}(t) \) is a moving volume, with surface \( \mathcal{S}(t) \), outward-pointing normal \( \hat{n}(x, t) \), and surface velocity \( v(x, t) \).

Problem 2  Using your result from Problem 1, show that the integral form of the conservation of mass for any control volume \( \mathcal{V} \) is
\[ \frac{d}{dt} \int_{\mathcal{V}(t)} \rho \, dV + \int_{\mathcal{S}(t)} \rho [(u - v) \cdot \hat{n}] \, dS = 0, \]
where \( \rho \) is the fluid density and \( u \) is the fluid velocity. HINT: The integral form of the conservation of mass for a material control volume \( \mathcal{V}_{\text{mat}} \), which moves with the fluid velocity \( u \), is
\[ \frac{d}{dt} \int_{\mathcal{V}_{\text{mat}}(t)} \rho \, dV = 0. \]

Problem 3  Recall that \( \tau_{ij} \) is the viscous stress tensor. From the second law of Thermodynamics it can be shown that \( \tau_{ij} S_{ij} \geq 0 \) must be true. Prove that
\[ \tau_{ij} S_{ij} \geq 0 \]
for a Newtonian fluid with
\[ \tau_{ij} = 2\mu S_{ij} + \lambda \nabla \cdot u \delta_{ij}. \]

Problem 4  In class we claimed that the contraction of a tensor symmetric in indices \( i \) and \( j \) with one that is anti-symmetric in the same indices is zero. Verify this. In other words, show that
\[ A_{ij} B_{ij} = 0 \]
where \( A_{ij} = A_{ji} \) and \( B_{ij} = -B_{ji} \).
Problem 5  State whether the following equations make correct or incorrect use of index notation. If the notation used is incorrect, briefly state why it is incorrect.

(a) \[
\frac{\partial u_i}{\partial t} = -\epsilon_{ijk} \frac{\partial u_k}{\partial x_j} - A_{ik} u_k - B_j \frac{\partial \tau_{ij}}{\partial x_j}
\]

(b) \[
\frac{\partial}{\partial t}(\omega_i \omega_j) = \omega_i \omega_k \left( \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) + 2\epsilon_{ijk} \frac{\partial}{\partial x^\ell} \left[ \nu \left( \frac{\partial u_k}{\partial x^\ell} + \frac{\partial u_\ell}{\partial x_k} - \frac{2}{3} \frac{\partial u_m}{\partial x_m} \delta_{kl} \right) \right]
\]

(c) \[
\frac{\partial}{\partial t}(\rho u_i u_i) = (u_j u_k) \frac{\partial u_j}{\partial x^k} + \epsilon_{kij} \left( \frac{\partial u_j}{\partial x^k} - \frac{\partial u_k}{\partial x^j} \right) + \mu \frac{\partial^2}{\partial x^k \partial x^k} (u_i u_i)
\]