HW #4

Due Thursday 18 October 2012 in class

Problem 1  In class we derived the non-dimensional form of the governing equations for a convection-dominated problem where the pressure scale was $\rho_\infty U_\infty^2$. At low Reynolds numbers the pressure is viscous dominated. Assuming the pressure scale is $\mu U_\infty / L$, derive the non-dimensional momentum equation for the incompressible flow of a fluid. What is the limiting equation as $Re_L \to 0$? Do not assume the flow is steady.

Problem 2  In class we derived the continuity equation in Cartesian coordinates to be

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0.$$ 

Using the integral form of the conservation of mass on the cylindrical control volume shown below, show that the equation of continuity in cylindrical coordinates $(r, \theta, z)$ is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho u_\theta) + \frac{\partial}{\partial z} (\rho u_z) = 0.$$ 

In doing so you will have shown that the divergence of a vector $\mathbf{v}$ in cylindrical coordinates is

$$\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (v_\theta) + \frac{\partial}{\partial z} (v_z),$$

which you’ll need for the next problem.

![Cylindrical volume element for Problem 2.](image1)

Problem 3  Using the fact that $\tau_{ij}$ is a tensor, show that the cylindrical components of $\tau_{ij}$ are

\[
\begin{align*}
\tau_{rr} &= 2\mu \frac{\partial u_r}{\partial r} + \lambda \nabla \cdot \mathbf{u} \\
\tau_{\theta\theta} &= 2\mu \left[ \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right] + \lambda \nabla \cdot \mathbf{u} \\
\tau_{zz} &= 2\mu \frac{\partial u_z}{\partial z} + \lambda \nabla \cdot \mathbf{u} \\
\tau_{r\theta} &= \mu \left[ \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} \left( \frac{u_r}{r} \right) \right] \\
\tau_{r\theta} &= \mu \left[ \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right] \\
\tau_{rz} &= \mu \left[ \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right]
\end{align*}
\]
Problem 4 Using your result from the above problem, show that the momentum equation in cylindrical coordinates is

\[
\rho \left( \frac{Du_r}{Dt} - \frac{\mathbf{u}_\theta^2}{r} \right) = \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial \tau_{rr}}{\partial \theta} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\tau_{r\theta} - \tau_{\theta\theta}}{r}
\]

\[
\rho \left( \frac{Du_\theta}{Dt} \right) + \frac{u_r \mathbf{u}_\theta}{r} = \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\tau_{r\theta}}{r}
\]

\[
\rho \frac{Du_z}{Dt} = \frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r}
\]

where

\[
D = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + u_\theta \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}
\]

Problem 5 Consider the steady laminar flow through the annular space formed by two coaxial tubes. The flow is along the axis of the tubes and is maintained by a pressure gradient \( dp/dz \), where \( z \) is along the axis of the tubes. Show that the axial velocity profile at any radius \( r \) is

\[
u_z(r) = \frac{1}{4\mu} \frac{dp}{dz} \left[ r^2 - a^2 - b^2 - a^2 \ln \left( \frac{b}{a} \right) \frac{r}{a} \right]
\]

where \( a \) is the radius of the inner tube and \( b \) is the radius of the outer tube. Find the radius at which the maximum velocity is reached, the volume rate of flow, and the stress distribution.

Problem 6 A long vertical cylinder of radius \( b \) rotates with angular velocity \( \Omega \) concentrically outside a smaller stationary cylinder of radius \( a \). The annular space is filled with fluid of viscosity \( \mu \). Show that the steady state velocity distribution is

\[
u_\theta = \frac{r^2 - a^2}{b^2 - a^2} \frac{b^2 \Omega}{r}
\]

Show that the torque exerted on either cylinder, per unit length, equals \( 4\pi \mu \Omega a^2 b^2 / (b^2 - a^2) \).