Interface evolves as: \[ z = \eta(x, y, t) \]

Equation of interface: \[ F(x, y, z, t) = z - \eta(x, y, t) \]

The unit normal of the interface is:

\[ \hat{n} = \frac{\nabla F}{|\nabla F|} \]

\[ \nabla F = \left( \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \right) \left[ z - \eta(x, y, t) \right] \]

\[ \nabla F = -\hat{e}_x \frac{\partial \eta}{\partial x} - \hat{e}_y \frac{\partial \eta}{\partial y} + \hat{e}_z \frac{\partial \eta}{\partial z} \]

\[ \nabla F = \left[ -\frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}, 1 \right]^T \]

\[ |\nabla F| = \left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 + \left( \frac{\partial \eta}{\partial y} \right)^2 \right]^{1/2} \]

Unit normal:

\[ \hat{n} = \left( -\frac{\partial \eta}{\partial x}, -\frac{\partial \eta}{\partial y}, 1 \right) \frac{1}{\left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 + \left( \frac{\partial \eta}{\partial y} \right)^2 \right]^{1/2}} \]
let $\bar{V}_{\text{int}}$ be the velocity of interface:

and $\bar{V}_{\text{int}} = [U_{\text{int}}, V_{\text{int}}, W_{\text{int}}]^T$

$$z = \eta (x, y, t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$\frac{\partial \eta}{\partial t} = \bar{V}_{\text{int}} - U_{\text{int}} \frac{\partial \eta}{\partial x} - V_{\text{int}} \frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} = \bar{V}_{\text{int}} \cdot \left( - \frac{\partial \eta}{\partial x} \hat{n}_x - \frac{\partial \eta}{\partial y} \hat{n}_y \right) + \bar{\epsilon}_z$$

$$\frac{\partial \eta}{\partial t} = \bar{V}_{\text{int}} \cdot \hat{n} = \frac{1}{\sqrt{1 + (\frac{\partial \eta}{\partial x})^2 + (\frac{\partial \eta}{\partial y})^2}}$$

The kinematic boundary condition for nonpermeable interface:

(2a) for liquid $\bar{u} \cdot \hat{n} = \bar{V}_{\text{int}} \cdot \hat{n}$

(2b) for gas $\bar{v} \cdot \hat{n} = \bar{V}_{\text{int}} \cdot \hat{n}$

If $\bar{u} = [U, V, W]$ and $\bar{v} = [V_g, V_g, W_g]^T$

From (1) and (2a)

$$\frac{\partial \eta}{\partial t} = - U \frac{\partial \eta}{\partial x} - V \frac{\partial \eta}{\partial y} + W$$

$$W = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} + V \frac{\partial \eta}{\partial y} \quad \text{(**) \quad}$$

Similarly from (1) and (2b)

$$\frac{\partial \eta}{\partial t} = - V_g \frac{\partial \eta}{\partial x} - V_g \frac{\partial \eta}{\partial y} + W_g$$

$$W_g = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} + V \frac{\partial \eta}{\partial y} \quad \text{(*** \quad}$$
Kinematic Boundary Condition

For liquid
\[
\frac{\partial \eta}{\partial t} \left[ 1 + \left( \frac{\partial \eta}{\partial y} \right)^2 + \left( \frac{\partial \eta}{\partial y} \right)^2 \right] \frac{\partial \eta}{\partial y} = \bar{u}_l \cdot \hat{n} \quad \text{at} \quad \frac{\eta}{\eta} = \frac{\eta}{\eta} (x, y, t)
\]

For gas
\[
\frac{\partial \eta}{\partial t} \left[ 1 + \left( \frac{\partial \eta}{\partial y} \right)^2 + \left( \frac{\partial \eta}{\partial y} \right)^2 \right] \frac{\partial \eta}{\partial y} = \bar{u}_g \cdot \hat{n} \quad \text{at} \quad \frac{\eta}{\eta} = \frac{\eta}{\eta} (x, y, t)
\]

5. Linearization

Assuming \( \left| \frac{\partial \eta}{\partial x} \right| \ll 1 \) and \( \left| \frac{\partial \eta}{\partial y} \right| \ll 1 \)

At \( \eta = \eta \),
\[
\frac{\partial \eta}{\partial t} \left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 + \left( \frac{\partial \eta}{\partial y} \right)^2 \right] \frac{\partial \eta}{\partial y} = \bar{u}_l \cdot \left[ -\frac{\partial \eta}{\partial x}, -\frac{\partial \eta}{\partial y}, 1 \right] \left[ 1 + \left( \frac{\partial \eta}{\partial x} \right)^2 + \left( \frac{\partial \eta}{\partial y} \right)^2 \right] \ll 1
\]

i.e. neglecting nonlinear term in \( \left( \frac{\partial \eta}{\partial x} \right)^2, \left( \frac{\partial \eta}{\partial y} \right)^2 \),

\( \Rightarrow \frac{\partial \eta}{\partial t} = \bar{u}_l \cdot \hat{n} \)

Similarly,
\[
\frac{\partial \eta}{\partial t} = -\bar{u}_g \frac{\partial \eta}{\partial x} - \bar{u}_g \frac{\partial \eta}{\partial y} + \bar{w}_g \quad \text{at} \quad \eta = \eta
\]

\( \Rightarrow \frac{\partial \eta}{\partial t} = \bar{u}_g \cdot \hat{n} \)

If \( |\eta| \ll 1 \) and interface has normal in \( z \)-direction,

\( \bar{u}_l \cdot \hat{n} = \bar{u}_l \cdot \hat{e}_z = \bar{w}_l \)

\( \bar{u}_g \cdot \hat{n} = \bar{u}_g \cdot \hat{e}_z = \bar{w}_g \)
\[ \frac{\partial \eta}{\partial t} = \omega_e \quad \text{at} \quad z = \eta \quad \text{for the liquid} \]

\[ \frac{\partial \eta}{\partial t} = \omega_g \quad \text{at} \quad z = \eta \quad \text{for the gas} \]

If \(|\eta| \ll 1\), we can assume up to first order that the interface is flat at \(z = 0\).

To verify, let us do a Taylor series expansion of \(\omega_e\) about \(z = 0\):

\[ \omega_e \big|_{z=\eta} = \omega_e \big|_{z=0} + \eta \frac{\partial \omega_e}{\partial z} \big|_{z=0} + \eta^2 \frac{\partial^2 \omega_e}{\partial z^2} \big|_{z=0} + \ldots \]

Can be neglected as \(|\eta| \ll 1\).

\[ \frac{\partial \eta}{\partial t} = \omega_e \quad \text{at} \quad z = 0 \quad \text{for liquid} \]

\[ \frac{\partial \eta}{\partial t} = \omega_g \quad \text{at} \quad z = 0 \quad \text{for gas} \]
The 2nd law of thermodynamics for the control volume $V$ bounded by surface $S = S_1 + S_2$

\[
\frac{d}{dt} \int_{V(t)} \varphi \, dV + \int_{S(t)} \left[ \frac{\partial \varphi}{\partial n} \right] n_j \, ds \geq 0 \quad (1)
\]

For material 1:
\[
\frac{d}{dt} \int_{V_1(t)} \varphi \, dV + \int_{S_1(t)+S_1'(t)} \left[ \frac{\partial \varphi}{\partial n} \right] n_j \, ds \geq 0 \quad (2)
\]

For material 2:
\[
\frac{d}{dt} \int_{V_2(t)} \varphi \, dV + \int_{S_2(t)+S_2'(t)} \left[ \frac{\partial \varphi}{\partial n} \right] n_j \, ds \geq 0 \quad (3)
\]

\(1 - 2 - 3 \geq 0\)

\[
\Rightarrow \quad \frac{d}{dt} \int_{V(t)} \varphi \, dV \geq \frac{d}{dt} \int_{V_1(t)} \varphi \, dV + \frac{d}{dt} \int_{V_2(t)} \varphi \, dV \quad (\text{interface does not generate or take away entropy})
\]

By construction, $S = S_1 + S_2 \Rightarrow \square = 1 - 1 + 1$

All integrals with $S$, $S_1$, $S_2$ will also cancel out.

\[
\int_{S(t)} \left[ \frac{\partial \varphi}{\partial n} \right] n_j \, ds = \int_{S_1(t)} \left[ \frac{\partial \varphi}{\partial n} \right] n_j \, ds + \int_{S_2(t)} \left[ \frac{\partial \varphi}{\partial n} \right] n_j \, ds
\]
\[ 0 \geq -\int_{S'(t)} \left[ \rho s (u_i - v_i) - \frac{\partial i}{T} \right] n_{j_1} dS - \int_{S'(t)} \left[ \rho s (u_i - v_i) - \frac{\partial i}{T} \right] n_{j_2} dS \]

By the 2nd law of thermodynamics.

\[ -\int_{S'(t)} \left[ \rho s (u_i - v_i) - \frac{\partial i}{T} \right] n_{j_1} dS \geq \int_{S'(t)} \left[ \rho s (u_i - v_i) - \frac{\partial i}{T} \right] n_{j_2} dS \]

Since \( A' \) is arbitrary and the inequality is true for any interface,

\[ -\int_{S'_{(1)}} \left[ \rho s (u_i - v_i) - \frac{\partial i}{T} \right] n_{j_{1'}} dS \geq \int_{S'_{(1)}} \left[ \rho s (u_i - v_i) - \frac{\partial i}{T} \right] n_{j_{2'}} dS \]

But \( n_{j_{1'}} = -n_{j_{2'}} \)

\[ \Rightarrow \left[ \rho s (u_i - v_i) - \frac{\partial i}{T} \right] n_{j_{1'}} \geq \left[ \rho s (u_i - v_i) - \frac{\partial i}{T} \right] n_{j_{2'}} \]

is the jump condition for entropy across the interface.

where \( n_{j_{2'}} \) is \( \frac{n_{j_{2'}}}{n_{j_{1'}}} \)
(4) Maxwell model to describe viscoelastic fluids: - 

e.g.: "Silly Putty"

\[
\dot{\sigma}_{ij} + \lambda \frac{\partial}{\partial t} \left( \sigma_{ij} \right) = 2 \eta \frac{\partial e_{ij}}{\partial t}
\]

\(\sigma\) = stress \\
\(e\) = strain with time \\
\(\eta\) = Rate of strain

(a) When the silly putty is squeezed slowly, the change in stress applied with time is very small, i.e. \(\frac{\partial}{\partial t} \sigma_{ij}\) is very small and we can neglect the contribution of this term in the equation.

\[\Rightarrow \dot{\sigma}_{ij} = 0\]

\[\Rightarrow \dot{\sigma}_{ij} \propto \epsilon_{ij}\]

We know that for a Newtonian fluid,

\[\sigma \propto \epsilon\]

\[\Rightarrow \dot{\sigma}_{ij} \propto \epsilon_{ij}\]

: When the putty is squeezed slowly, such that \(\frac{\partial}{\partial t} \sigma_{ij}\) is very small, it satisfies the constitutive law for a Newtonian fluid. : It behaves like a liquid.

(b) When the putty is rolled into a ball and dropped, the rate of change of stress with time is very large when it hits the wall/surface, \(\frac{\partial}{\partial t} \sigma_{ij}\) becomes a dominant term in the equation.

\[\sigma_{ij}\] is very small compared to the large change in stress per unit time, when the putty hits the ball.

\[\Rightarrow \lambda \frac{\partial}{\partial t} \sigma_{ij} = 2 \eta \frac{\partial e_{ij}}{\partial t}\]

\[ \text{Rate of stress} \propto \text{Rate of strain} \]
\[ \Rightarrow \quad \text{stress} \propto \text{strain}. \]

This is Hooke's law of solids that undergo stress.

: When the putty is rolled into a ball & dropped, it satisfies Hooke's law and behaves like a solid.

\[ \text{due to large rate of change of stress with time,} \]
\[ \text{when it hits the surface (i.e. during impact).} \]