Problem 1  Kinematic boundary condition at a free surface. In class we derived the kinematic boundary condition that the normal velocity, \(u_n\), is continuous across any material interface. Consider the specific case of a liquid-gas interface where the surface separating the two fluids is \(z = \eta(t, x, y)\). If the velocity field on the liquid side of the interface is \(u_L\) and the velocity field on the gas side of the interface is \(u_G\), derive the kinematic boundary condition, one for the fluid and one for the gas, in terms of the derivatives of \(\eta\). Your answer should look something like \(u_L \cdot n = f(\partial \eta / \partial t, \partial \eta / \partial x, \partial \eta / \partial y, u_L)\) where \(f\) is to be determined. You will need to determine \(n\), the unit normal, from \(\eta\).

Problem 2  Your answer to Problem 1 is called the *nonlinear* kinematic condition between the dependence of \(n\), the unit normal, on the surface evolution \(\eta\). It is common to linearize the boundary condition by assuming that the interface stays nearly flat, which amounts to assuming \(|\partial \eta / \partial x| \ll 1\) and \(|\partial \eta / \partial y| \ll 1\). If, in addition, \(|\eta| \ll 1\) and the undisturbed interface has a normal in the \(z\)-direction, write down the linearized kinematic boundary condition.

Problem 3  In class we derived the “jump” conditions across any interface as

\[
\begin{align*}
[\rho(u_j - v_j)]_1 n_i &= [\rho(u_j - v_j)]_2 n_j \\
[\sigma_{ij} - \rho u_i (u_j - v_j)]_1 n_j &= [\sigma_{ij} - \rho u_i (u_j - v_j)]_2 n_j \\
[\sigma_{ij} u_i - q_j - \rho E (u_j - v_j)]_1 n_j &= [\sigma_{ij} u_i - q_j - \rho E (u_j - v_j)]_2 n_j
\end{align*}
\]

where \(n_i\) is the unit normal pointing from side 2 into side 1 and \(\sigma_{ij} = -p \delta_{ij} + \tau_{ij}\) is the total stress tensor.

Following the same procedure as developed in class, use the second law of thermodynamics

\[
\frac{d}{d t} \int_{V(t)} \rho s \, dt + \oint_{S(t)} [\rho (u_j - v_j) + q_j / T] n_j \, dS \geq 0
\]

to derive the “jump” condition for the entropy, \(s\),

\[
[\rho s(u_j - v_j) + q_j / T]_1 n_j \geq [\rho s(u_j - v_j) + q_j / T]_2 n_j.
\]
Problem 4  The simplest equation for describing a fluid that is both viscous and elastic is the Maxwell model

\[ \tau_{ij} + \lambda_1 \frac{\partial}{\partial t} \tau_{ij} = 2\eta_0 S_{ij} \]

where \( \tau_{ij} \) is the stress tensor and \( S_{ij} \) is the rate-of-strain tensor. \( \lambda_1 \) and \( \eta_0 \) are constants.

“Silly putty” is a material that has this constitutive law. Using this constitutive law, explain why (in words)

(a) When the putty is slowly squeezed, it behaves like a liquid.

(b) When the putty is rolled into a ball and dropped, it behaves like a solid.