HW #2
Due Tuesday 25 September 2012 in class

Problem 1  Consider a velocity field

\[ u_i = K \frac{x_2 \delta_{i1} - x_1 \delta_{i2}}{x_1^2 + x_2^2} \]

where \( K \) is a positive constant.

(a) Plot (do not sketch) the streamlines associated with the velocity field.

(b) Sketch the motion a unit square with points \{(1, 1), (3/2, 1), (3/2, 3/2), (1, 3/2)\} will take.

(c) Compute the velocity gradient tensor \( \partial u_i / \partial x_j \)

(d) Compute the rate-of-strain and spin tensors \( S_{ij} \) and \( \Omega_{ij} \)

(e) To what physical object does this velocity field correspond?

Problem 2  Repeat Problem 1 but for the velocity field \( u_i = y \delta_{i1} \). Omit subquestion (e).

Problem 3  In class we derived equations for mass, momentum, and (total) energy conservation. For no body forces those equations are, respectively,

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_j} = 0 \]

\[ \frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \rho u_i u_j + p \delta_{ij} - \tau_{ij} \right) = 0 \]

\[ \frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x_j} \left[ (\rho E + p) u_j + q_j - u_i \tau_{ij} \right] = 0. \]

Using these equations derive the following in index notation

(a) The equation for the kinetic energy \( \rho u_i u_i / 2 \) per unit volume.

(b) The equation for the internal energy \( \rho e \) per unit volume.

(c) The equation for the entropy \( \rho s \) per unit volume. Recall the Gibbs equation \( T \, ds = de + p \, dp^{-1} \).
**Problem 4**  In problem 3 you found the equation for the entropy. In that equation you should have found a term like

\[ \tau_{ij} \frac{\partial u_j}{\partial x_i}. \]

This term is called the *dissipation function* and it represents the viscous work going into deformation.

(a) Show that \( \tau_{ij} \frac{\partial u_j}{\partial x_i} = \tau_{ij} S_{ij} \) where \( S_{ij} \) is the rate-of-strain tensor.

(b) For a Newtonian fluid, show that \( \Phi = \tau_{ij} S_{ij} \geq 0. \)

(c) In addition, show that \( \Phi \geq 0 \) only when \( \mu \geq 0 \) and \( 3\lambda + 2\mu \geq 0. \)

**Problem 5**  Recall that the vorticity is defined as

\[ \omega_k = \epsilon_{ijk} \frac{\partial u_j}{\partial x_i}. \]

Using the momentum equation given in Problem 2, derive the governing equation for the vorticity of the form

\[ \frac{\partial \rho \omega_i}{\partial t} + \cdots = 0. \]

**Problem 6**  Using index notation, show, for any field \( f \), that

\[ \frac{\partial \rho f}{\partial t} + \frac{\partial}{\partial x_j} (\rho f u_j) = \rho \left( \frac{\partial f}{\partial t} + u_j \frac{\partial f}{\partial x_j} \right) \]

for a fluid without any sources of mass.