Solutions to Midterm #2
15 May 2014

Instructions: You have 3 hours to complete this exam. Show all work and use the supplemental material provided. You will be primarily graded on your demonstration of logic in solving the problems.

Printed Name: 

Date: 

Student code: I agree to abide by the University of Illinois Student Code. I attest that the work on this exam is my own.
Sign: 
Problem 1  A Ludwieg tube is a special wind tunnel that is used to create high-speed flows in a test section, as shown in the figure. The tube works by breaking a diaphragm separating a high-pressure driver gas (with pressure and temperature $p_4$ and $T_4$, respectively) from a low-pressure driven gas (with pressure and temperature $p_1$ and $T_1$). From your shock-tube work you know that a shock travels into the driven section and the expansion fan travels into the driver section. This expansion wave propagates into a converging-diverging nozzle and starts a high-speed flow that moves to the right. To first approximation, the nozzle pressure ratio is equal to $p_4/p_3$, where $p_3$ is the pressure just after the expansion fan tail.

Using your knowledge of shock tubes and quasi-1-D flow, answer the following questions. The nozzle is designed for Mach 6 flow and the fluid is air-air with an uniform initial temperature of 300 K.

(a) (5 pts) The nozzle area ratio $A_n/A_t$ that is consistent with the nozzle design Mach number
(b) (10 pts) The nozzle pressure ratio $p_4/p_3$ that is consistent with the nozzle design Mach number
(c) (50 pts) The required pressure ratio $p_4/p_1$ that must be in place to create the expansion fan that travels through the nozzle and starts the flow. (Hint: you must calculate the shock Mach number $M_s$ to find this.)
(d) (35 pts) To make this problem feasible ‘on paper’, we assumed that the nozzle pressure ratio is $p_4/p_3$. What is wrong with this assumption? How would you fix it?
Workspace for problem 1

(a) **(5 pts) Answer** The design Mach number of 6 corresponds to the supersonic solution of the area-Mach number relation

\[
\left( \frac{A}{A^*} \right)^2 = \frac{1}{M^2} \left( \frac{2}{\gamma + 1} \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \right)^{\frac{\gamma}{\gamma - 1}}
\]

assuming steady, inviscid, isentropic quasi-one-dimensional flow. Since the nozzle is choked, \( A^* = A_t \). Looking up the value of \( M = 6 \) in NACA Report 1135 gives the answer as

\[
\frac{A_n}{A_t} = 53.18
\]

(b) **(10 pts) Answer** Consistent with (a), the nozzle design point is isentropic and the gas in region 4 is at rest; hence, \( p_4 \) is a total pressure and \( p_3 \) is the nozzle exit pressure. Thus we can use the isentropic formula

\[
\frac{p_4}{p_3} = \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma}{\gamma - 1}} = \frac{p_4}{p_3}
\]

or look the value up in the NACA 1135 tables. Either way, the answer is

\[
\frac{p_4}{p_3} = 1.57888
\]

(c) **(50 pts) Answer** According to the problem statement, we need to find the pressure ratio \( p_4/p_1 \) that has the right value to give us the necessary nozzle pressure ratio \( p_4/p_3 \) such that a Mach 6 flow is established over the nozzle with area ratio \( A_n/A_t \). Since a Ludwieg tube is nothing more than a special shock tube, we can use our shock tube results. The overall pressure ratio is related to \( p_4/p_3 \) by

\[
\frac{p_4}{p_1} = \frac{p_4}{p_3} \frac{p_2}{p_1}
\]

which simplifies to

\[
\frac{p_4}{p_1} = \frac{p_4}{p_3} \frac{p_2}{p_1}
\]

since the pressure is continuous over the contact surface. However, \( p_2/p_1 \) is unknown and is related to the shock, with Mach number \( M_s \), created when the diaphragm bursts.

The ratio \( p_4/p_3 \) is also related to shock since it is created by the expansion wave that results from the burst diaphragm. In class we showed that this ratio can be found from the method-of-characteristics to be

\[
\frac{p_1}{p_4} = \left[ 1 - \frac{\gamma_4 - 1}{2} \frac{u_3}{a_4} \right]^{\frac{\gamma_4}{\gamma_4 - 1}}.
\]  \( \text{(1)} \)

If \( u_3 \) is known then, because the velocity is also continuous across the contact surface, we know \( u_2 \), the velocity of the gas in the lab frame after the incident shock. When the flow velocity \( u_2 \) is known, then we can find the shock Mach number \( M_s \) that created it and, hence, get \( p_2/p_1 \).

Solving for \( u_3 \) is Eq. (1) yields

\[
u_3 = \frac{2a_4}{\gamma_4 - 1} \left[ 1 - \left( \frac{p_3}{p_4} \right)^{\frac{\gamma_4 - 1}{\gamma_4}} \right]^{\frac{1}{\gamma_4 - 1}}
\]

\[
= \frac{2 \cdot 347.13}{1.4 - 1} \left[ 1 - \left( \frac{1}{1.57888} \right)^{\frac{1.4 - 1}{1.4}} \right]^{\frac{1}{1.4 - 1}}
\]

\[
= 1,129.53 \text{ m/s}.
\]  \( \text{(2)} \)

If the pressure ratio is unknown, then we can use the isentropic formula from (b) to find it. If the pressure ratio is known, then we can use the shock tube relations to find the shock Mach number and, hence, get \( p_2/p_1 \).
Thus, $u_2 = 1,129.53 \text{ m/s}$ and we need to find the shock Mach number that generates this post-shock velocity in the lab frame. If $U_s$ is the shock velocity in the lab frame, then a simple coordinate transformation from the shock-to-lab frame shows that

$$u_2 = U_s \left( 1 - \frac{u'_2}{u'_1} \right)$$

$$= \frac{a_1}{\gamma_1} \left( \frac{p_2}{p_1} - 1 \right) \sqrt{\frac{2\gamma_1}{\gamma_1+1} \left( \frac{p_2}{p_1} + \frac{\gamma_1-1}{\gamma_1+1} \right)}$$

(4)

where $u'_2/u'_1$ is the velocity ratio across the shock in the shock frame. Solving Eq. (4) for $p_2/p_1$ gives 19.89 which, from the NACA 1135 Report gives a shock Mach number of 4.15 to the nearest entry.

Putting it all together gives the pressure ratio of

$$\frac{p_4}{p_1} = \frac{p_1}{p_1} \cdot \frac{p_2}{p_1}$$

$$= 1,578.88 \cdot 19.89$$

$$= \sqrt{31,403.92} \approx 31,000$$

which is the answer desired. The pressure ratio of $p_4/p_1 \approx 31,000$ is high but not out of reach of several Ludwieg tubes available across the globe.

(d) **Answer** The answer from (c) used two major components: an expansion wave into the driver section and a shock wave into the driven section. For the shock wave, it travels in a constant area portion of the Ludwieg tube and, consequently, our analysis tools are still valid. For the expansion wave, however, it travels through the nozzle and therefore sees a local area $A(x)$ that changes; our expansion wave results assumed that the area was constant. So our use of Eq. (1) is incorrect.

To fix things, we would need to rederive the unsteady method-of-characteristics to allow for a quasi-one-dimensional area change. This isn’t difficult but it not needed for this exam. You can show, for example, that the quantity

$$u \pm \frac{2c}{\gamma - 1}$$

is no longer constant in general but varies as $A(x)$ changes.
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