Problem 1  In class we saw that when small compressibility disturbances were present in one-dimensional motion the speed-of-sound, $c$, defined as

$$c = \sqrt{\left. \frac{\partial p}{\partial \rho} \right|_s},$$

was an inherent velocity scale. In this definition the derivative of pressure with respect to density is taken at constant entropy, i.e., at isentropic conditions. With this introduction, please do the following:

(a) Show that the equation of state $p = P(\rho, s)$ can be written for a calorically perfect ideal gas as

$$\frac{p}{p_1} = e^{(s-s_1)/C_v}\left(\frac{\rho}{\rho_1}\right)^{\gamma}$$

where $\{p_1, s_1, \rho_1\}$ are the pressure, entropy, and density at a reference condition and $C_v$ is the specific heat at constant volume.

(b) Use the result from (a) to show that

$$\left. \frac{\partial p}{\partial \rho} \right|_s = \gamma \frac{p}{\rho}.$$

(c) Suppose instead, as Newton did, that the speed of sound was based on an isothermal derivative, $c_t = \sqrt{\left. \frac{\partial p}{\partial \rho} \right|_T}$. What is the new formula for $c_t$ and how does it differ from the original one $c$?

(d) Why, then, should the sound speed be computed based on the isentropic derivative of pressure with respect to density, rather than the isothermal one? The answer has to do with the rate at which the gas is compressed by the acoustic wave. As a model for acoustic compression, consider the figure below of a closed cylindrical container of diameter $D$ air held in a frictionless box that allows heat transfer to and from the ambient air and that has a movable top lid. The height of the uncompressed box is $L$ and the amount of compression is $\ell(t) = \epsilon(1 + \sin \omega(t - \pi/2))$, where $\epsilon \ll \ell$ is a small distance. Assuming that the thermodynamic state within the container is constant and that the momentum, kinetic energy, and gravitational forces within it is negligible, show that

(1) The density satisfies $\rho(t) = \rho_0 L/(L - \ell(t))$.

(2) The energy equation reduces to

$$\frac{d}{dt}(\rho e V) = p(t) \ell A + \int_S \dot{q} \cdot n \, dS,$$

where $A = \pi D^2/4$ is the cross-sectional area.

(3) For the heat flux, we assume that the air within the container can exchange heat with the ambient according to the formula $\dot{q} = -\kappa(T(t) - T_a)n$ where $T_a$ is the ambient temperature, and $\kappa$ is the thermal conductivity per unit length. Note that $S$ is a function of time and is equal to $2A + (L - \ell)\pi D$. Using this information, derive the ordinary differential equation for $T(t)$ in terms of the geometry and no other thermodynamic quantity.
(4) Use Matlab’s ode45 function to numerically solve for \( T(t) \) assuming the following information: \( L = 1 \text{ cm}, \) \( \epsilon = 0.1 \text{ mm}, T_0 = 300 \text{ K}, p_0 = 101 \text{ kPa}, \omega = 600\pi, T_a = 300 \text{ K}, \kappa = 0.024 \text{ W/(m}^2\text{.K)}, C_v = 714 \text{ J/(kg.K)}. \) All of these values correspond to air and an oscillation frequency of 300 Hz.

(5) Repeat your solution to (4) with \( \kappa = 0; \) that is, neglect the heat transfer.

(6) Compare your solutions to (4) and to (5). Comment on what you see and whether assuming sound is an isentropic process.

Problem 2  A reusable hypersonic vehicle being considered by the USAF is to fly horizontally through the atmosphere at an altitude of 100 km. For a range of Mach numbers from 4 to 10, plot the maximum temperature that the vehicle’s exterior skin will feel as a function of Mach number. (The U.S. Standard Atmosphere (1976) is defined in NASA Technical Manual 74335 and is available free online from NASA. Look in Table 1, page 68, to get the atmospheric conditions at 100,000 m.) For reference, indicate the melting point of ‘standard’ Aluminum (933 K) and ‘standard’ titanium (1933 K) and indentify at which Mach numbers Aluminum and titanium are no longer viable materials for use in the vehicle skin without thermal protection. You may assume that \( \gamma = 1.4. \)

Problem 3 Using the one-dimensional normal shock relations, derive the following expression for the Mach number after the shock in terms of the incoming Mach number \( M_1, \)

\[
M_2^2 = \frac{M_1^2 + 2/(\gamma - 1)}{[2\gamma/(\gamma - 1)]M_1^2 - 1}
\]

Problem 4  Consider a normal shock wave in air. The upstream conditions are given by \( M_1 = 3, p_1 = 1 \text{ atm, and } \rho_1 = 1.23 \text{ kg/m}^3. \) Calculate the downstream values of \( p_2, T_2, \rho_2, M_2, u_2, p_0, \text{ and } T_0. \) Draw the process on a Mollier diagram using the correct values for \( h \text{ and } s. \) Anderson 3.4, with modification.

Problem 5  A pitot tube is mounted on an aircraft that can travel with variable speed from \( M_a = 0.1 \) to \( M_b = 2.0 \) at a height of 1 km. As a function of the Mach number \( M, \) plot the ratio of the stagnation to static pressure, \( p_0/p, \) that the pitot tube would measure. From this ratio, estimate the velocity of the aircraft assuming (a) incompressible flow and (b) isentropic relation. What do your results tell you about pitot tubes operating in a supersonic stream?

Problem 6  Air at a pressure of 105 kPa, a temperature of 20 °C, and a Mach number of 3 passes through a normal shock. Assume air to be a perfect gas.

(a) What are the velocities of the gas upstream and downstream of the shock wave?

(b) How much does the pressure rise through the shock wave?

(c) What would the pressure after the shock be if the velocity had been reduced isentropically from the initial velocity to that behind behind the shock?

Problem 7  The pressure and density ratios across a shock wave in a perfect gas are related by the Rankine-Hugoniot relationship.
(a) Show that this relationship is
\[
\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1} \frac{p_2}{p_1} + 1
\]
\[
\frac{\gamma + 1}{p_2} + 1
\]
(b) What is the maximum density ratio for air ($\gamma = 1.4$) as the pressure ratio $p_2/p_1$ becomes very large?
(c) For very small pressure differences (a weak shock) show that this expression becomes equal to the isentropic relationship $p^{-1}dp = \gamma \rho^{-1}d\rho$. 