Stability of a ML

After linearization, Laplace transform in t, Fourier transform in x:

\[ \hat{\eta}(s,k) = \frac{\text{complicated term}}{\Delta(s,k)} \]

\[ \Delta(s,k) = \rho_1 \left\{ gk - (s + ikU_1)^2 \right\} - \rho_2 \left\{ gk + (s + ikU_2)^2 \right\} \]

We seek the zeros of \( \Delta(s,k) = 0 \) [solve for s as func of k] \( s^* \Rightarrow \Delta(s^*,k) = 0 \)

\[ s^* = -ik \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \frac{\sqrt{\frac{k^2 \rho_1 \rho_2 (U_2 - U_1)^2}{(\rho_1 + \rho_2)^2} - \frac{kg(\rho_1 - \rho_2)}{\rho_1 + \rho_2}}}{\rho_1 + \rho_2} \]

Identify instability when \( \text{Re}\{s^*_f\} \) is positive. Need \(\frac{k^2 \rho_1 \rho_2 (U_2 - U_1)^2}{(\rho_1 + \rho_2)^2} > \frac{kg(\rho_1 - \rho_2)}{\rho_1 + \rho_2} \)
Now, let's interpret this result for our motivating Air-over-H₂O(2) system:

\[ P_1 = \frac{1}{1000} P_2, \quad U_2 = 0 \quad \text{plug in} \quad U_1^* \geq \frac{99.05}{\sqrt{k}} \quad \text{Condition for instability.} \]

Recall that \( \lambda = \frac{2\pi}{k} \) is the wavelength. So we can write the above as:

\[ U_1^* \geq 39.5 \sqrt{\lambda} \quad \text{Condition for instability.} \]

\[
\begin{array}{c|c}
\lambda & U_1^* \\
1 \text{ mm} & 1.24 \text{ m/s} \\
1 \text{ cm} & 3.95 \text{ m/s} \\
1 \text{ m} & 39.5 \text{ m/s} \\
\end{array}
\]

For this model, short waves are more unstable and will likely be observed first.

It turns out that including surface diffusion is important:

\[ P_2 = P_1 + \frac{2\pi}{k^2} \quad \text{Surface diffusion "damps out" small wavelength waves.} \]

\[ \frac{2\pi}{k^2} \quad \text{Surface diffusion} \]

\[ R \approx \text{radius of curvature.} \]

If we repeat the analysis with \( \lambda > 0 \Rightarrow \min U_1^* \approx 6.6 \text{ m/s.} \)
Other cases $\Delta(s,k) = 0$:

1) Surface gravity wave: $p_1 < p_2$, $u_1 = u_2 = 0$

   $s^* \propto \pm i \sqrt{gk}$ — purely imaginary $\implies$ just oscillate.

2) Internal gravity wave: $u_1 = u_2$, $p_1 \neq p_2$

   $s^* = \pm \sqrt{\frac{g(p_1 - p_2)}{p_1 + p_2}} \quad \rightarrow$ instability if $p_1 > p_2$ (heavier fluid over light fluid)

   Rayleigh-Taylor.

3) Instability due to shear: $p_1 = p_2$, $u_1 \neq u_2$

   $s^* = -\frac{1}{2} i k (u_1 + u_2) \pm \frac{1}{2} k (u_2 - u_1)$

   always has a instability when $u_1 \neq u_2$!

   Kelvin-Helmholtz instability.

   $\implies$ Shear layers are always unstable.
Recap the semester:

1) molecules → continuum
2) flow kinematics
3) constitutive modeling
4) boundary conditions
5) Exact 1-D flows/heat transfer (unsteady)
   - Steady
   - Self-similar
   - Falkner–Skan
   - Integral (Lighthill/Prandtl)
6) BLs → Blasius
7) MLS/Shear layers, & 3D wall jets, laminar jets
8) Buoyancy - free & convective
9) Flow stability