October 27, 2020

- Finish Falkner-Skan: \( u(x,y) = U_e(x)f(\eta), \eta = \frac{y}{8ax} \)
- Start Integral Bls.

\[
\begin{align*}
\beta & = 0.1988 \quad \text{(sep.)} \\
U_e(x) & = Kx^m \\
\delta(x) & = \frac{2}{\sqrt{m+1}} U_e(x)
\end{align*}
\]

The outcome of self-similarity:

Pressure gradient parameter:

(inviscid flow model)

\[
\eta \rightarrow \infty \quad f' \rightarrow 1 \\
U \rightarrow U_0(x)
\]

Self-similar solution:

\[
f'' + ff' + \beta (1-f'^2) = 0, \quad \beta = \frac{2m}{m+1}
\]
To interpret F.S. on a "real-life" object, we need to extract the m (or β) parameters. How to get m?

One approach is to fit the profile $Kx^m$ to the real $U_e(x)$ in a region.

Require that $S_u$ is continuous between segs.
Another approach is to compute a local $m(x)$.

\[
U_e = K x^m \\
U_e' = K m x^{m-1}
\]

\[
\frac{i}{U_e} U_e' = m/x \implies m(x) = \frac{x U_e'}{U_e}
\]
Recall: Prandtl identified the BL with the BL equations.

1908 Blasius

1930 Falkner-Skan

Pohlhausen \{ looked for an approximate solution from the beginning \} (1920)

Thwaits simplified the Pohlhausen method to make it useful (1945)

exact solutions

approximate methods.
Fact: The best approximate BL methods are based on integrals of the BL equations.

We first have to derive the integral BL equations:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}
\]

Replace \(-\frac{1}{\rho_0} \frac{\partial p}{\partial x} \approx \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}\)

Freestream momentum.

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}.
\]

Replace \(\rho_0 \frac{\partial u}{\partial y} \approx 1\)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho_0} \frac{\partial \rho}{\partial y}.
\]
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + \frac{1}{\rho_0} \frac{\partial \tau}{\partial y} \tag{\star} \]

Multiple continuity \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \) by the quantity \((u-u_e)\) and subtract it from \((\star)\).

The result is

\[ \int_{-\infty}^{\infty} - \frac{1}{\rho_0} \frac{\partial \tau}{\partial y} \, dy = \frac{\partial}{\partial t} (u_e - u) + \frac{\partial}{\partial x} (u u_e - u^2) + (u_e - u) \frac{\partial u_e}{\partial x} + \frac{\partial}{\partial y} (\sigma u_e - \sigma u) \]

\[ \frac{T_u}{P_\infty} = \frac{\partial}{\partial t} \int_{0}^{\infty} (u_e - u) \, dy + \frac{\partial}{\partial x} \int_{0}^{\infty} (u u_e - u^2) \, dy + \frac{\partial u_e}{\partial x} \int_{0}^{\infty} (u_e - u) \, dy + \left[ \sigma (u_e - u) \right]_0^\infty \]

so we are left with

\[ \frac{T_u}{P_\infty} = \frac{\partial}{\partial t} \int_{0}^{\infty} (u_e - u) \, dy + \frac{\partial}{\partial x} \int_{0}^{\infty} (u u_e - u^2) \, dy + \frac{\partial u_e}{\partial x} \int_{0}^{\infty} (u_e - u) \, dy \]

\( \\text{since} \ \nu = 0 \text{ at } \infty \)

\( u = u_e \text{ at } \infty \)
We have seen terms like these before:

\[ \delta^u = \text{displacement thickness} = \int_0^\infty (1 - \frac{u}{U_e}) dy. \]

\[ \Theta = \text{momentum thickness} = \int_0^\infty \frac{u}{U_e} (1 - \frac{u}{U_e}) dy \]

Then the equation

\[ \frac{T_w}{P_\infty} = \frac{\partial}{\partial t} \int_0^\infty (U_e - u) dy + \frac{\partial}{\partial x} \int_0^\infty (U_e u - u^2) dy + \frac{\partial U_e}{\partial x} \int_0^\infty (U_e - u) dy \]

becomes, after dividing by \(P_\infty U_e^2\),

\[ \frac{T_w}{P_\infty U_e^2} = \frac{1}{U_e^2} \frac{\partial}{\partial t} (U_e \delta^u) + \frac{\partial \Theta}{\partial x} + (2 \Theta + \delta^u) \frac{dU_e}{dx} \frac{1}{U_e} \]

\[ \frac{\delta^u}{\Theta} = H = \text{shape factor}. \]
The final form is

\[
\frac{C_f}{2} = \frac{1}{u_e^2} \frac{2}{\partial t} (u_e S^*) + \frac{d \theta}{dx} + (2 + H) \frac{\theta}{u_e} \frac{du_e}{dx}
\]

Von Kármán Integral Momentum equation.

Note that \( C_f \), \( S^* \), and \( \Theta \) are all unknowns.

Note also that if \( u_e \) = const (and steady), then

\[
\frac{C_f}{2} = \frac{d \theta}{dx}
\]
How do we solve this? We'll focus on the steady state problem:
\[
\frac{1}{2} \alpha_f = \frac{d \theta}{dx} + (2 + H) \frac{\theta}{\alpha_e} \frac{d \alpha_e}{dx}
\]

Three unknowns: \( \alpha_f, \theta, \delta^* \) (\( H = \delta^*/\theta \)). Pohlhausen guessed the velocity profile:
\[
\frac{u(x,y)}{\alpha_e} = f(\eta) = a\eta + b\eta^2 + c\eta^3 + d\eta^4, \quad \eta = \frac{y}{\delta(x)}
\]
Pohlhausen used these 4 conditions to find \( \{a, b, c, d\} \):

1. \( y = \delta \): \( u = \alpha_e, \ d\alpha_p/\alpha_e = 0, \ d^2\alpha/\alpha_e^2 = 0 \)

2. \( y = 0 \): evaluated the differential BL equation on the wall:
\[
\frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = -\frac{1}{\rho_0} \frac{d \rho_p}{dx} + \nu_0 \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{4-th cond.}
\]
The coefficients \( a, b, c, d \) may be found to be:

\[
a = 2 + \frac{\Lambda}{6}, \quad \Lambda = \frac{\delta^2}{\nu_{oo}} \frac{dW_e}{dx}
\]

\[
b = -\frac{\Lambda}{2}
\]

\[
c = -2 + \frac{\Lambda}{2}
\]

\[
d = 1 - \frac{\Lambda}{6}
\]

So the Pohlhausen profile

\[
u(x,y) = \frac{U(x,y)}{U_e} = F(\eta) + \Lambda G(\eta)
\]

\[= \left(2\eta - 2\eta^3 + \eta^4\right) + \frac{\Lambda}{6} \left(\eta - 3\eta^2 + 3\eta^3 - \eta^4\right)
\]
To find $\delta$, use the Von Kármán integral equation:

$$\delta^* = \int_0^\infty 1 - \frac{u}{u_e} \, dy = \delta \left( \frac{3}{10} - \frac{L}{120} \right)$$

$$\Theta = \int_0^\infty \frac{u}{u_e} \left( 1 - \frac{u}{u_e} \right) \, dy = \cdots = \delta \left( \frac{3L}{215} - \frac{L}{945} - \frac{L^2}{9072} \right)$$

$$\tau_w = \mu_{oo} \frac{du}{dy} \bigg|_w = \cdots = \frac{\mu_{oo} u_e}{\delta} \left( 2 + \frac{L}{6} \right)$$

We can plug these into the VKI equation and get a mess!

Pohlhausen found that re-writing the VKI equation with $\Theta$, not $\delta$, was simpler. The result is

$$\frac{1}{2} u_e \frac{d}{dx} \left( \frac{\Theta^2}{V_{oo}} \right) + [2 + f(\lambda)] \lambda = g(\lambda)$$
\[
\frac{1}{2} \rho e \frac{d}{dx} \left( \frac{\theta^2}{v_\infty^2} \right) + \left[ 2 + f(\chi) \right] \lambda = g(\chi)
\]

**where:** \[
\lambda = \frac{\theta^2}{v_\infty^2} \frac{d\rho e}{dx}
\]

\[
f(\chi) = 2 + 4.14 (0.25 - \chi)^2 - 83.5 (0.25 - \chi)^3 + 854 (0.25 - \chi)^4 + 4576 (0.25 - \chi)^5
\]

\[
g(\chi) = (\lambda + 0.09)^{0.62}
\]

\[
\tau_\infty = \frac{\mu_\infty \rho e}{\theta} g(\chi)
\]

\[
\delta^* = \theta f(\chi)
\]

difficult to solve.
Thwaites took experimental data of $\lambda, f(\lambda), g(\lambda)$ and observed that

\[
\frac{U_e}{V_\infty} \frac{d(\theta^2)}{dx} \sim 0.45 - 6\lambda
\]

Thwaites’ observation means that

\[
\frac{U_e}{V_\infty} \frac{d\theta^2}{dx} + \left[2 + f(\lambda)\right] \lambda = g(\lambda)
\]

Can be replaced by

\[
\frac{U_e}{V_\infty} \frac{d\theta^2}{dx} = 0.45 - 6\lambda
\]
Using \( \lambda = \frac{\Theta^2}{\nu_0} \frac{dU_e}{dx} \), the Thwaites approximation to Pohlhausen is

\[
\frac{U_e}{\nu_0} \frac{d\Theta^2}{dx} = 0.45 - \frac{6\Theta^2}{\nu_0} \frac{dU_e}{dx}
\]

which has the solution

\[
\Theta^2(x) = \frac{0.45 \nu_0}{[U_e(x)]^6} \int_0^x \left[ U_e(s) \right]^5 ds
\]

(assumes \( \Theta(0) \approx 0 \)) Thwaites-Pohlhausen.

Once you have \( \Theta(x) \), then you know

\[
\lambda = \frac{\Theta^2}{\nu} \frac{dU_e}{dx}, \quad f(\lambda), \quad g(\lambda) \Rightarrow \tau_w = \frac{\mu_0 U_e}{\Theta} g(\lambda), \quad S^* = \Theta f(\lambda)
\]
We can compare to Blasius:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Thwaites - Pohlhausen</th>
<th>Blasius</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\Delta x}{2}$</td>
<td>$1.75$</td>
<td>$1.732$</td>
</tr>
<tr>
<td>$\frac{8\text{eq}}{x}$</td>
<td>$5.84$</td>
<td>$4.9$</td>
</tr>
<tr>
<td>$\frac{\theta}{x}$</td>
<td>$0.686$</td>
<td>$0.664$</td>
</tr>
<tr>
<td>$C_f$</td>
<td>$0.686$</td>
<td>$0.664$</td>
</tr>
</tbody>
</table>

Pretty good