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Last time: \( \nabla \cdot \mathbf{u} = C \ldots \) \( \sim \) convection (at Mach #)

+ \( C \ldots \) \( \sim \) unsteady

+ \( C \ldots \) \( \sim \) viscous effects

+ \( C \ldots \) \( \sim \) heat transfer / thermal expansion effects

Scenario: consider a low Mach \# \( \ll 1 \), convection-dominated flow, that is "slow" to evolve.

\[ M \ll 1 \sim \text{neglect terms that are proportional to } \mathbf{M}, \mathbf{M}^2 \]

This combination of assumptions tells us that \( \nabla \cdot \mathbf{u} \sim \Theta(M^2) \approx 0 \)
One we say that \( \nabla \cdot \mathbf{u} \approx 0 \) and we have a convection-dominated flow, we get a simpler set of equations to work with:

\[
\frac{\partial \rho}{\partial t} + \sum_{j} \frac{\partial}{\partial x_j} (\rho u_j) = 0
\]

Now also assume that all streamlines come in with the same density \( \approx \rho_0 = \text{constant everywhere} \):

Go to momentum equation

\[
\rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \mu \frac{\partial u_i}{\partial x_j} \right] + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_i}{\partial x_j \partial x_j} \approx 0
\]

\\
\frac{\partial \mu}{\partial x_j} = \text{gradient of viscosity} = \frac{\partial}{\partial x_j} \left( \frac{\partial \rho}{\partial x_j} \right)

\text{How "big" is this term? Convection dominated flow } \Rightarrow T_0^* - T^* \approx \Theta(M^2)

\Rightarrow \frac{\partial T}{\partial x_j} \approx \frac{H^2}{L} \ll 1

\Rightarrow \text{length scale } \approx L^*

\Rightarrow \text{neglect } \frac{\partial \mu}{\partial x_j} \approx \Theta(H^2)
The momentum equation is now
\[ \rho_\infty \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = - \frac{\partial p}{\partial x_i} + \mu(T) \frac{\partial^2 u_i}{\partial x_j \partial x_j} \]

\[ \mu(T) = \mu(T_\infty) + \left[ \mu(T) - \mu(T_\infty) \right] \]

\[ \text{t.s. approxim.} \quad \mu(T) - \mu(T_\infty) \sim \mu(T_\infty) \frac{\partial \mu}{\partial T} (T - T_\infty) \]

\[ \Rightarrow [\mu(T) - \mu(T_\infty)] \sim \mathcal{O}(H_{\beta}^2) \ll 1 \]

\[ \Rightarrow \mu(T) \approx \mu(T_\infty) = \mu_\infty + \mathcal{O}(H_{\beta}^2) \quad \text{error} \]

Plug into momentum to get
\[ \rho_\infty \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = - \frac{\partial p}{\partial x_i} + \mu_\infty \frac{\partial^2 u_i}{\partial x_j \partial x_j} \]

\[ (\nabla^2 u_i)_i \]
For the energy equation, we saw last time that

\[ \rho C_p \left( \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} \right) = \left( \frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} \right) + \tau_{ij} S_{ij} - \frac{\partial \tilde{g}_i}{\partial x_j} \approx \Theta(M^4) \]

so our energy equation for a low-Mach-number, convection-dominated flow is

\[ \rho \infty C_p \left( \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} \right) = - \frac{\partial \tilde{g}_i}{\partial x_j} = - \frac{\partial}{\partial x_j} \left( -k \frac{\partial T}{\partial x_j} \right) \]

\[ \approx \left( \frac{\partial k}{\partial x_j} \right) \frac{\partial T}{\partial x_j} + k \frac{\partial^2 T}{\partial x_j \partial x_j} \]

\[ \frac{\partial k}{\partial x_j} = \frac{dk}{dt} \frac{\partial T}{\partial x_j} \approx \Theta(M^4) \Rightarrow \frac{\partial k}{\partial x_j} \approx 0. \] So now our energy equation becomes

\[ \rho \infty C_p \left( \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} \right) = k \frac{\partial^2 T}{\partial x_j \partial x_j} \]
\[ \nabla \cdot \vec{u} = 0, \quad \rho = \rho_\infty \text{ (constant)} \]

\[ \rho_\infty \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial P}{\partial x_i} + \mu_\infty \frac{\partial^2 u_i}{\partial x_j \partial x_j} \]

\[ \rho_\infty C_p \left( \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} \right) = \kappa_\infty \frac{\partial^2 T}{\partial x_j \partial x_j} \]

Low-Mach-number, convection-dominated flow

\( \Rightarrow \) energy is one-way coupled to momentum

\( \Rightarrow \) solve for \( u_j \) first, then solve for \( T \)
Why do we solve the energy equation at all?

Convection-dominated: $T_0 - T \sim O(H^2)$

Why not just assume $T = T_0$ (a constant) — we could, but we lose out on understanding how "heat" (energy) is transported in low-speed flows.
Exact Solution #1 - Steady Couette flow.

\[ \mathbf{U}, T \rightarrow \infty \]

Predict the velocity profile \( \mathbf{u} \) and the temperature profile \( T \).

\textbf{Assumptions}

1) Steady

2) Constant properties in \( x \)-direction

3) No pressure gradient \( \Rightarrow \) requiring the fluid move entirely due to plate motion.

Low-Mach-number, convective-dominated flow \( \Rightarrow \rho \approx \rho_0 \)

\( \Rightarrow \nabla \cdot \mathbf{u} = 0 \)

\( \Rightarrow \rho_0 \left( \frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_j} \right) = - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_i \partial x_j} \)

\( \Rightarrow \rho_0 c_p \left( \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} \right) = -k \frac{\partial^2 T}{\partial x_j \partial x_j} \)

\( \approx 0 \) since no variation in \( x \)

Start with \( \nabla \cdot \mathbf{u} = 0 \):

\( \frac{\partial u_i}{\partial x_i} = 0 \Rightarrow \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \Rightarrow \text{vertical velocity} = \text{constant} = 0 \)

because of no-penetration on parallel sides.

\( \Rightarrow u_i(x_1, x_2) = U(x_2) \delta_{i2} \)

\( \Rightarrow \mathbf{u} = U(y) \hat{e}_x \)
Since $u_2 = 0$, we don't need to consider the 2-momentum equation. The 1-momentum equation is

$$p_{\infty} \left( \frac{\partial u_1}{\partial t} + u_j \frac{\partial u_1}{\partial x_j} \right) = -\frac{\partial p_1}{\partial x_1} + \mu_{\infty} \frac{\partial^2 u_1}{\partial x_1 \partial x_1}$$

$$p_{\infty} \left( \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} \right) = \mu_{\infty} \left( \frac{\partial^2 u_1}{\partial x_1 \partial x_1} + \frac{\partial^2 u_1}{\partial x_2 \partial x_2} \right)$$

$$0 = \mu_{\infty} \left( \frac{\partial^2 u_1}{\partial x_2 \partial x_2} \right)$$

Trivial to solve:

$$u_1(y) = A + B y \Rightarrow u_1(y) = U \cdot \left\{ \frac{1}{2} \left( 1 + \frac{y}{U} \right) \right\}$$

or

$$\frac{u_1}{U} = \frac{1}{2} \left( 1 + \frac{y}{U} \right)$$