Last time:

\[ 
\sigma_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} 
\]

\[ 
= 2\mu \sigma_{ij} + \lambda \nabla \cdot \mathbf{u} \delta_{ij} 
\]

"Shear"  "Volume change"

The volume change appears in both terms, but is hidden in the first:

\[ 
\sigma_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) 
\]

\[ 
\text{tr} \sigma_{ij} = \sigma_{ii} = \frac{\partial u_i}{\partial x_i} 
\]

People prefer to separate out the volume change from the purely shear part by introducing the deviatoric stress tensor:
\[ \tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \frac{\partial^2 u_k}{\partial x_k^2} s_{ij} \]

\[ = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{2}{3} \frac{\partial u_k}{\partial x_k} s_{ij} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} s_{ij} \right) + \lambda \frac{\partial^2 u_k}{\partial x_k^2} s_{ij} \]

\[ = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} s_{ij} \right) + \left( \lambda + \frac{2\mu}{3} \right) \frac{\partial^2 u_k}{\partial x_k^2} s_{ij} \]

\[ 2S_{ij} \Rightarrow S_{ii} = 0 \]

\[ \tau_{ii} = 2\mu S_{ij} + \mu_B \nabla \nabla \cdot \nabla S_{ij} \]

Convenient

1) Incompressible flow: \( \nabla \cdot \vec{u} = 0 \Rightarrow \tau_{ii} = 2\mu S_{ij} \)

2) "Stokes' hypothesis" \( \Rightarrow \mu_B \equiv 0 \Rightarrow \lambda = -\frac{2}{3}\mu \)

\( \mu_B \) bulk viscosity.

\begin{tabular}{|c|c|}
\hline
Fluid & \( \mu_B/\mu \) \\
\hline
He & 0 \\
Ar & 0 \\
H_2 & 32 \\
O_2 & 0.4 \\
N_2 & 0.8 \\
CO_2 & 1000 \\
"Air" & 0.6 \\
H_2O & 3.1 \\
\hline
\end{tabular}

only true for monatomic gases.
Heat transfer: We will define the thermal conductivity as

\[ \dot{q}_i = -K_{ij} \frac{\partial T}{\partial x_j} \]

\[ = -K_{ij} \frac{\partial T}{\partial x_j} + \beta_{ijk} \frac{\partial T}{\partial x_k} \frac{\partial T}{\partial x_j} + \ldots \]

Experiments have shown that \( \dot{q}_i \propto \frac{\partial T}{\partial x_i} \Rightarrow K_{ij} \) is diagonal.

If the fluid is isotropic, then

\[ K_{ij} = k \delta_{ij} \Rightarrow \text{Fourier' \"Law\"} \]

\[ \Rightarrow \dot{q}_i = -k \frac{\partial T}{\partial x_i} \quad (k > 0) \]

For second law compatibility.
How do we predict $\mu, M_b, k$? The physical origin of viscosity and "heat" conduction depends on the molecular description of the fluid. The molecular transport of momentum $\rightarrow \mu, M_b$ and energy $\rightarrow k$ are called transport coefficients.

We will pursue a simple description $(\mu, M_b, k)$ in terms of simple molecular motion.

$\vec{a}(x_2)$ \quad [\vec{a} = \text{velocity, temperature, ...}]

Goal: quantify the rate at which the property $\vec{a}$ passes across $x_2 = x_{20}$ per unit area.

$\nabla \cdot \vec{a} \rightarrow \nabla \cdot \vec{a}$, the flux of $\vec{a}$.

- Let's assume that all molecules within a distance $s x_2$ cross the surface in a given time interval $\Delta t$.

- Let's assume that the average velocity of these molecules is $\vec{v}$.

- Let's assume that there are $n$ particles per unit volume.

\[ n \vec{a} = \frac{n \vec{v}}{2} \left[ \vec{a}(x_{20} - s x_2) - \vec{a}(x_{20} + s x_2) \right] \quad \text{proportionality factor.} \]
We note that $\delta x_2 = a \lambda$ where $\lambda$ is the mean free path.

Expand $\Delta \tilde{\alpha}$ in a Taylor series, we get

$$\Delta \tilde{\alpha} \approx -\eta \tilde{c} \eta \alpha_{\Gamma} \frac{d \tilde{\alpha}}{d x_2} \mid_{x_2 = x_{2_0}}$$

For a gas indistinguishable particles:

- $\tilde{\alpha} = \text{average (bulk) velocity of a bunch of particles}$
- $\alpha_{\Gamma} = \text{molecular mass}$
- $\tilde{c} = \text{internal energy per unit mass}$

$$- \Delta \tilde{\alpha} = -\beta \tilde{\alpha} \tilde{c} \frac{d \tilde{\alpha}}{d x_2} \mid_{x_2 = x_{2_0}} \Rightarrow \mu = \beta \tilde{\alpha} \tilde{c} \lambda$$

Functions of $T$ only:

$$\mu = \beta \tilde{\alpha} \tilde{c} \lambda$$

$$\Delta \tilde{\alpha} = -\beta \tilde{\alpha} \tilde{c} \lambda \frac{d T}{d x_2} \mid_{x_2 = x_{2_0}}$$
Sutherland's Law for viscosity:

\[
\frac{\mu(T)}{\mu(T_0)} = \left( \frac{T}{T_0} \right)^{3/2} \frac{T_0 + S\mu}{T + S\mu}, \quad S\mu \text{ is a const.}
\]

For liquids, modeling \( \mu \) is very hard. Experiments for estimate \( \mu(T) \).

Example:

\[
\ln \frac{\mu}{\mu_0} = a\mu + b\mu \left( \frac{T_0}{T} \right) + c\mu \left( \frac{T_0}{T} \right)^2
\]

Node: geo: \( \mu \uparrow \) as \( T \uparrow \) \( \text{why?} \) \n
liquid: \( \mu \downarrow \) as \( T \uparrow \)
Boundary Conditions

- What are Boundary Conditions? - Encode how a fluid interacts with its surroundings.

- Most famous: no-slip.

\[ \mathbf{u} \cdot \hat{n} = 0 \quad \text{on the boundary.} \]

is an observation that if the fluid device isn't "too small" and if the fluid density isn't "too small" then

\[ \mathbf{u} \cdot \hat{n} = \mathbf{v} \cdot \hat{n} \]

there is no proof of no-slip, and there are cases where it is not true. Ex: spacecraft at high altitudes, associated with non-continuum effects.

There are many papers discussing the physical origins of no-slip.

⇒ How do we derive a BC in general? (for a continuum)

Use conservation of mass, momentum, and energy across surfaces separating two different fluids/two different states of matter.