September 8, 2020

- Last time: vector form of deformation.
  concluded that $\varepsilon_{ij}$ can only depend on $\frac{\partial u_i}{\partial x_j}$

- Today: tensorial form & examples

Recall that if we have two points, $\vec{x}$ and $\vec{x} + \Delta \vec{x}$, then the velocity difference can be written

$$d\vec{u} = \vec{u}(\vec{x} + \Delta \vec{x}) - \vec{u}(\vec{x})$$

can be written

$$du_i = \frac{\partial u_i}{\partial x_i} \Delta x_k$$

VGT

$$= \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k} \right) \Delta x_k$$

= Strain rank

rotation

$$= \left[ S_{ji} + \Omega_{ij} \right] \Delta x_k$$
We also showed that

\[ \Omega_{lm} = \frac{1}{2} \epsilon_{iem} \omega_{i} \Leftrightarrow \omega_{e} = \epsilon_{ijk} \frac{\partial \Omega_{kj}}{\partial x_{j}} \]

\[ \omega_{e} = \epsilon_{ijk} \Omega_{jk} \]

We can then write the relative velocity as

\[ du_{i} = \left( S^{ai} + \Omega_{ai} \right) dx_{a} \]

\[ = S^{k}_{ai} dx_{k} + \frac{1}{2} \epsilon_{jli} \omega_{j} dx_{k} \]

\[ 2^{nd} \text{ term: } \left( \frac{1}{2} \varepsilon^{k}_{ij} x_{k} \right) \]

\[ = \{ \text{shear deformation} \} + \{ \text{solid body rotation} \} \]
Example: \[ \dot{\mathbf{u}} = \mathbf{u}_0 + \dot{\vartheta} \times \mathbf{x} \]

\[ \uparrow \text{ translational} \]

\[ \epsilon \text{ solid body rotation about the origin} \]

\[ u_i = u_{0i} + \epsilon_{jki} \dot{\vartheta}_j x_k \]

Wrong.

\[ S_{ki} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right] = \frac{1}{2} \frac{\partial}{\partial x_k} \left( u_{0i} + \epsilon_{jki} \dot{\vartheta}_j x_k \right) + \frac{1}{2} \frac{\partial}{\partial x_i} \left( u_{0k} + \epsilon_{jkl} \dot{\vartheta}_j x_l \right) \]

\[ = \frac{1}{2} \frac{\partial}{\partial x_k} \left( u_{0i} + \epsilon_{jki} \dot{\vartheta}_j x_k \right) + \frac{1}{2} \frac{\partial}{\partial x_i} \left( u_{0k} + \epsilon_{jkl} \dot{\vartheta}_j x_l \right) \]

\[ = \frac{1}{2} \epsilon_{jki} \dot{\vartheta}_j \frac{\partial x_k}{\partial x_i} + \frac{1}{2} \epsilon_{jkl} \dot{\vartheta}_j \frac{\partial x_l}{\partial x_i} \]

Question: \[ \frac{\partial x_k}{\partial x_i} = ? \rightarrow \delta_{ik} \]

\[ = \frac{1}{2} \epsilon_{jki} \dot{\vartheta}_j \delta_{ik} + \frac{1}{2} \epsilon_{jkl} \dot{\vartheta}_j \delta_{li} \]

\[ = 0 \]
\[ \omega_{ki} = \frac{1}{2} \left( \frac{\partial \omega_i}{\partial x_k} - \frac{\partial \omega_k}{\partial x_i} \right) \]

only difference from Sui definition.

\[ \Omega = \begin{bmatrix} 0 & -\dot{\theta}_3 & \dot{\theta}_2 \\ \dot{\theta}_3 & 0 & -\dot{\theta}_1 \\ -\dot{\theta}_2 & \dot{\theta}_1 & 0 \end{bmatrix} \]
Example: Consider the velocity field \( u_i = U x_2 \delta_{i2} \uparrow \)
\[ \vec{u} = U x_2 \hat{e}_x \]

\[
\begin{align*}
\nabla \cdot \vec{u} &= \frac{\partial u_i}{\partial x_k} - \frac{\partial}{\partial x_k} \left\{ U x_2 \delta_{i2} \right\} \\
&= U \delta_{i2} \frac{\partial x_2}{\partial x_k} \\
&= U \delta_{i2} \delta_{k2}
\end{align*}
\]

\[
S_{ki} = \frac{1}{2} U \left( \delta_{i1} \delta_{k2} + \delta_{i2} \delta_{k1} \right)
\]

\[
\Omega_{ki} = \frac{1}{2} U \left( \delta_{i1} \delta_{k2} - \delta_{i2} \delta_{k1} \right)
\]


\[
S = \begin{bmatrix} 0 & U/2 & 0 \\ U/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0 & -U/2 & 0 \\ U/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[ \frac{1}{2} \omega_3 \text{ clockwise rotation about the } x_3 \text{-axis.} \]

The simple linear velocity profile is the sum of uniform shear in the \( x_1 \)-\( x_2 \) plane + uniform rotation about the \( x_3 \) axis.
Why did we start with $\sigma_{ji} = -p \delta_{ji} + t_{ji}$ and end with $VGT = S_{ki} + D_{ki}$?

→ need to create a relationship that connects the stresses to the rate-of-deformation.

→ need to find how $\sigma_{ji}$ depends on $VGT$.

\[ \tau_{ji} = f(VGT) \]

\[ \tau_{ji} = \alpha \frac{\partial u_i}{\partial x_j} \text{ or } \tau_{ji} = \alpha \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_j} \]

We need some help. A fact: the most general linear relationship between 2nd-order tensors is

\[ \tau_{ij} = \alpha_{ijkl} \frac{\partial u_k}{\partial x_l} \]

More general form:

\[ \tau_{ij} = \alpha_{ijkl} \frac{\partial u_k}{\partial x_l} + \beta_{ijklmn} \frac{\partial u_k}{\partial x_l} \frac{\partial u_m}{\partial x_n} + \ldots \]
We classify fluids according to

\[ T_{ij} = \bar{\alpha}_{ijkl} \frac{\partial u_k}{\partial x_l} \]

1) if \( \bar{\alpha}_{ijkl} \) is independent of the VGT = Newtonian.

2) otherwise, the fluid is non-Newtonian.

In general, \( \bar{\alpha} \) can depend on the \( P \), the \( T \), and the VGT.
Consider a simple shear rheometer

\[ \text{By construction, the velocity is} \quad u_i = U \left( \frac{y}{h} \right) \]

The simple shear rheometer is built to create a

\[ u_i = U(x_2) \delta_{i2} \quad \Rightarrow \quad \nabla \cdot \mathbf{u} = \frac{\partial u_k}{\partial x_2} = \frac{dU}{dy} \delta_{k1}, \delta_{k2} \]

Measure the shear on the bottom face, \( \tau_{21} \):

\[ \tau_{21} = \sigma_{21} \delta_{21} \frac{\partial u_k}{\partial x_2} = \sigma_{21} \frac{dU}{dx_2} \]

\[ \Rightarrow \quad b \mu \]

\[ \frac{dU}{dy} \]

\[ \mu \]
The coefficient $\mu$ is called the shear viscosity coefficient and it is a transport property. In general,

$$\mu = f(\rho, T, \partial u_i / \partial x_j)$$

**Neutonian fluid:** $\mu = f(\rho, T)$ only.

**Non-Neutonian fluid:** $\mu = f(\rho, T, \partial u_i / \partial x_j)$

**Measure of shear:** $\dot{\gamma} = \sqrt{\dot{S}_{ij} \dot{S}^{ij}}$

**Measure of stress:** $\tau = \sqrt{T_{ik} T^{ik}}$
Examples of different kinds of fluids

Newtonian: air, water, alcohol.

Shear-thinning fluid: blood, paint

Shear-thickening: corn starch & water (oblice), butter.

In this class we will concentrate on Newtonian fluids

\[ \mu = f(p, T) \] and independent of the VGT.

Under this assumption, the most general form of the relation

\[ \tau_{ij} = \delta_{ij} \frac{d\mu}{dx_i} \]

is

most general

\[ 3^4 = 81 \] components.
To make progress, we will assume that the fluid is isotropic (the shear stress cannot depend on the orientation of the fluid element).

**Tensor fact:** The most general isotropic fourth order tensor is

\[ \sigma_{ijkl} = \mu \delta_{ik} \delta_{jl} + \mu' \delta_{il} \delta_{jk} + \lambda \delta_{ij} \delta_{kl} \]

3 independent coefficients.

For \( \tau_{ij} = \tau_{ji} \) to be true, it must be that \( \mu = \mu' \)

\[ \tau_{ij} = \left\{ \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) + \lambda \delta_{ij} \delta_{kl} \right\} \frac{\partial u_k}{\partial x_i} \]

\[ \tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \frac{\partial u_k}{\partial x_i} S_{ij} \]

\[ \tau_{ij} = 2\mu S_{ij} + \lambda \nabla \cdot \vec{u} S_{ij} \] — Galilean invariant \( \checkmark \)