August 27, 2020

Today

+ Piazza reminder — sign up!
+ Tensors and summation (index) notation. — see H.O. on website for more details.
+ Surface & body forces <-> stress tensor.

From last time
Cartesian Tensors and Summation (Index) Notation

- Summation convention:

\[ \overrightarrow{u} = u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3 \]

\[ = \sum_{i=1}^{3} u_i \hat{e}_i \]

\[ \| \hat{e}_i \| = 1, \text{ for } i = 1, 2, 3 \]

\[ = u_i \hat{e}_i \]

**Rules**

1. Any index that appears only once, is not summed, and is called the free index:

\[ u_j, \quad j = 1, 2, \text{ or } 3 \]
2. No index may appear more than twice in the same term.

\[ A_{ij} B_{jk} = A_{i1} B_{1k} + A_{i2} B_{2k} + A_{i3} B_{3k} \]

\( A_{ij} B_{ij} \) **not allowed!**

3. We call a repeated index a **dummy index** and we are free to change its identity, so long as we don't violate rule #2:

\[ \vec{a} \cdot \vec{b} = \left( \sum_{i=1}^{3} a_i \hat{e}_i \right) \left( \sum_{j=1}^{3} b_j \hat{e}_j \right) \]

\[ = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \hat{e}_i \cdot \hat{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

\[ = \sum_{i=1}^{3} a_i b_i = a_1 b_1 = a_2 b_2 = a_3 b_3 \]
4. Each term in an expression must have the same free indices:

\[ a_i = b_i + c_i \quad \checkmark \]
\[ a_i = b_i + c_j \quad \times \quad \text{wrng.} \]

Another example that is wrong is

\[ a_{ji} = b_{kl} \quad \times \]

5. Free indices can be changed if done consistently:

\[ a_i = D_{ij}b_j + E_{ij}c_j \quad \checkmark \]
\[ \rightarrow \quad k \]
\[ a_k = D_{kj}b_j + E_{kj}c_j \quad \checkmark \]
Two special "tensors"

1. Kronecker Delta. The $\delta_{ij}$

\[ \delta_{ij} = \hat{e}_i \cdot \hat{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

Warning: $\delta_{ij}$ as defined above is specific to Cartesian coordinate systems.
2. Alternating "tensor" / Levi-Civita symbol

\[ \varepsilon_{ijk} = (\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_k = \begin{cases} 
0 & \text{if any } (ijk) \text{ are same} \\
+1 & \text{if } (ijk) \text{ are cyclic} \\
-1 & \text{if } (ijk) \text{ are anti-cyclic} 
\end{cases} \]

\[ \varepsilon_{223} = 0 \]

\[ \varepsilon_{423} = \varepsilon_{231} = \varepsilon_{312} = 1 \]

\[ \varepsilon_{521} = \varepsilon_{213} = \varepsilon_{132} = -1 \]
Basic Vector Operations Using Index Notation

1. Dot product:
\[ \vec{a} \cdot \vec{b} = (a_i \hat{e}_i) \cdot (b_j \hat{e}_j) = a_i b_j \hat{e}_i \cdot \hat{e}_j \]
\[ = a_i b_j \delta_{ij} \]
\[ = a_i b_i = a_j b_j \]
2. Cross-product:
\[
\mathbf{\hat{a} \times \hat{b}} = (\mathbf{\hat{a} \times \hat{b}})_k \mathbf{\hat{e}_k}
\]
\[
= \left\{ \left[ (a_i \mathbf{\hat{e}_i}) \times (b_j \mathbf{\hat{e}_j}) \right] \cdot \mathbf{\hat{e}_k} \right\} \mathbf{\hat{e}_k}
\]
\[
= a_i b_j \left\{ (\mathbf{\hat{e}_i} \times \mathbf{\hat{e}_j}) \cdot \mathbf{\hat{e}_k} \right\} \mathbf{\hat{e}_k}
\]
\[
\overset{\varepsilon_{ijk}}{\longrightarrow}
\]
\[
= a_i b_j \varepsilon_{ijk} \mathbf{\hat{e}_k}
\]

So, the \(k\)th component of \(\mathbf{\hat{a} \times \hat{b}}\) is \(a_i b_j \varepsilon_{ijk}\)

(useful to remember.)
Facts

1. Gradient of a scalar, $\phi$:
\[
\nabla \phi = \frac{\partial \phi}{\partial x_i} \hat{e}_i
\]

2. Divergence of a vector, $\vec{a}$:
\[
\nabla \cdot \vec{a} = \frac{\partial a_i}{\partial x_i}
\]

3. Curl of a vector, $\vec{a}$:
\[
\nabla \times \vec{a} = \frac{\partial a_j}{\partial x_i} \mathbf{\epsilon}_{ijk} \hat{e}_k 
\rightarrow (\nabla \times \vec{a})_k = \frac{\partial a_j}{\partial x_i} \mathbf{\epsilon}_{ijk}
Surface Forces i.e., the Stress Tensor

Goal of this class: predictive (quantitative) theory of fluid dynamics.

- Molecular description of a fluid
- Statistical quantum mechanics
- Dynamic equations for the molecules
- Dynamic equations for the continuum
  - Averaging
  - Thermo.
  - Transport coefficients
  - \( \mu, \kappa, \cdots \)

- Constitutive model
- Quantitative theory

Not covered in AE412 ME411
We start by assuming that there are two types of force on a fluid element:

\[ p \mathbf{F}(x,t) \delta V \]

- force per unit mass.

1. Body forces: created by long-range forces (e.g., gravity, electrostatics)
2. **Surface Force**: There are short-range forces that rapidly decay away from a surface. Typically, the range is $\sim 1-10$ intermolecular spacings.

Examples:
- friction (momentum exchange)
- surface tension
- pressure

Must assume that the $\{\text{penetration depth}\} \ll \{\text{size of our fluid element}\}$.
We will work with a convenient element

\[ \hat{n} = \text{unit outward normal} \]

The surface force we will write as \( \sum_i(\vec{x}, \hat{n}, t)\delta A \)

The dimension of \( \sum_i \) is \( \text{Force / area} \).
Convention:

\[ \hat{n} \quad \text{SV} \quad \hat{n} \quad \hat{\Sigma} \quad \Rightarrow \quad \text{the fluid element is in tension.} \quad \hat{\Sigma} = A\hat{n}, \quad A > 0 \]

Consider two adjacent fluid elements that share a common face.

\[ \text{SV}_1 \quad \text{SV}_2 \quad \text{FBD} \quad \hat{\Sigma}_1(x, \hat{n}_1, t) \quad \hat{n}_2 \quad \hat{\Sigma}_2(x, \hat{n}_2, t) \]

same face.
We know that \( \hat{\Sigma}_1 + \hat{\Sigma}_2 = 0 \). Thus:

\[
\hat{\Sigma}_1(\bar{x}, \hat{n}_1, t) = -\hat{\Sigma}_2(\bar{x}, \hat{n}_2, t)
\]

note that \( \hat{n}_1 = -\hat{n}_2 \)

\[
\hat{\Sigma}_1(\bar{x}, \hat{n}_1, t) = -\hat{\Sigma}_2(\bar{x}, -\hat{n}_1, t)
\]

or

\[
\hat{\Sigma}(\bar{x}, \hat{n}, t) = -\hat{\Sigma}(\bar{x}, -\hat{n}, t)
\]

\( \Rightarrow \hat{\Sigma}_1 \) is an odd function of \( \hat{n} \)
We can because it tells us how we are allowed to model $\Sigma$:

$$\Sigma_i(x, \hat{n}, t) = a(x, t)\hat{n}_i + \beta_{ijk}(x, t)\hat{n}_j \hat{n}_k + ...$$

must be zero.

Recall the tetrahedral element, sum the surface forms

$$\delta A \vec{\Sigma}(x, \hat{n}, t) + \delta A_1 \vec{\Sigma}(x_a, \hat{n}_a, t) + \delta A_2 \vec{\Sigma}(x_b, \hat{n}_b, t) + \delta A_3 \vec{\Sigma}(x_c, \hat{n}_c, t)$$
We will use the fact that no $SOV$ is small and perform a Taylor-Series Expansion about a point $\bar{x}_0$, the centroid of the element.